# On the Operational Nature of Causality

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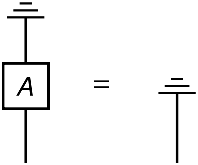
# Philosophical preamble

- R.B Lindsay: “[The operationalist] usually does not deny that it is possible to define physical concepts by other means than by physical operations. But he insists it is only by means of operationalism that we can hope to avoid inconsistencies and contradictions”.
- Bridgman: “If we sufficiently extend our range we shall find that nature is intrinsically and in its elements neither understandable nor subject to law”

# What is causality in an operational theory?



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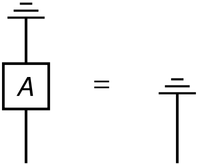
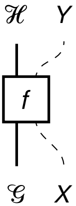


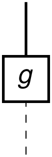
A mathematical statement about the terminality of the tensor unit of process theories, does it have a clear operational description?

The various $CPM$, $CP^*$ (related to $C^*$ algebras and quantum logic) constructions offer a diagrammatic perspective of the algebraic structure of QT but at the expense of the direct physical interpretation of the morphisms.

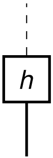
# What is causality in CPTs?

Categorical probabilistic theories are an attempt to recoincile the framework of OPTs (explicitly based on axioms with a concrete physical inspirations) and the clean categorical foundations and diagrammatic calculus of the CPM construction.



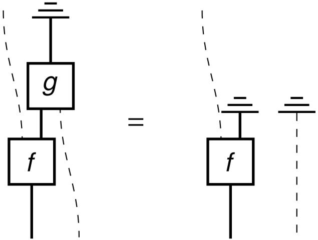


preparation



observation

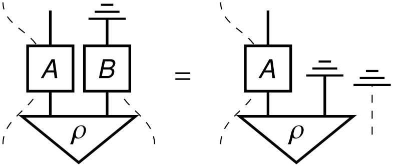
# No-signalling from the future



*Causality is the observation that the local output of the test f is independent of the input of the following process g.*

# Causality means localisation

An analogous way to represent the axiom of causality is in terms of the well definedness of the operational meaning of localisation.



# How can we therefore classify indefinite causality?

Assuming an underlying SMC, everything we can talk about is 'circuits'. Definite causal order already lurks in the mathematical language used to describe the theory: implied by its categorical structure and the additional assumption of no signalling from the future.

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How can we trascend this limitation and talk about definite and *indefinite* causality in an operational and theory independent way?

# We can use presheaves...

Presheaves allow us to formalise the concept of variable set.

$$\mathscr{E}\colon \mathscr{C} \longrightarrow \mathrm{Set}^{\mathrm{op}}$$

Functoriality makes sure that for any morphism in $\mathscr{C}$, $U \subseteq V$ there is a restriction function $\rho^V_U\colon \mathscr{E}(V) \longrightarrow \mathscr{E}(U)$ satisfying:

$$\rho^V_U \circ \rho^W_V = \rho^W_U, \quad \rho^U_U = id_{\mathscr{E}(U)}$$

## Compatible family

Let $\{U_i\}_i$ be a collection of objects of $\mathscr{C}$ then a compatible family for $\{U_i\}_i$ is a collection of sections $s_i \in \mathscr{E}(U_i)$ such that:

$$\rho^{U_i}_{U_j \cap U_i}(s_i) = \rho^{U_j}_{U_j \cap U_i}(s_j)$$

## Sheaves

Informally, a functor $\mathscr{E}$ is a *sheaf* when all compatible sections can be uniquely glued togheter.

# …to represent 'a topological structure' of causal sets into correlations

## Causal scenario

A causal scenario is a tuple $\Sigma := (\Omega, \underline{I}, \underline{O})$ where:

- $\Omega$ is a causal set, i.e. a directed acyclic graph
- $\underline{I} = (I_\omega)_{\omega\in\Omega}$ and $\underline{O} = (O_\omega)_{\omega\in\Omega}$ are families of non-empty finite sets;

## Locale of causally restricted inputs

$$\mathcal{O}_\Sigma := \coprod_{\lambda\in\Lambda(\Omega)} \prod_{\omega\in\lambda} \left(\mathscr{P}(I_\omega)_{\setminus\{0\}}\right)$$

equipped with the following partial order:

$$(V,\mu) \leq (U,\lambda) \iff \mu \subseteq \lambda \text{ and } \forall\omega\in\mu.\ \ V_\omega \subseteq U_\omega$$

Let $\Sigma = (\Omega, \underline{I}, \underline{O})$ be a causal scenario. The sheaf of sections $\mathcal{E}_\Sigma$ is the functor $\mathcal{O}_\Sigma \to \mathrm{Set}^{\mathrm{op}}$ defined as follows. To each object $(U, \lambda)$ of $\mathcal{O}_\Sigma$, the functor assigns the following set of functions:

$$\mathcal{E}_\Sigma(U, \lambda) := \left\{ f : \prod_{\omega \in \lambda} U_\omega \to \prod_{\omega \in \lambda} O_\omega \middle| f \text{ is causal } \right\}$$

$$\mathcal{E}_\Sigma((V, \mu) \leq (U, \lambda)) := f \mapsto f|_{\coprod_{\omega \in \mu} V_\omega}$$

We say that a function $f$ is causal if for any two $x, y \in \lambda$ we have that $x \not\leq y$ implies that the output value $f(\underline{i})_y \in O_y$ at event $y$ does not depend on the input value $i_x$ at the event $x$:

$$\forall \underline{i}, \underline{j} \in \prod_{\omega \in \lambda} U_\omega . \left[ \left( \forall z \in \lambda_{\setminus \{x\}}.\ \ i_z = j_z \right) \implies f(\underline{i})_y = f(\underline{j})_y \right]$$

# A theory independent framework for causality

## Presheaf of distributions

Let $\Sigma = (\Omega, \underline{I}, \underline{O})$ be a causal scenario, the presheaf of $\mathbb{R}^+$ distributions for $\Sigma$ is the presheaf $\mathscr{D}_{\mathbb{R}^+}\mathscr{E}_\Sigma$ obtained by composing the $\mathbb{R}^+$-distribution monad $\mathscr{D}_{\mathbb{R}^+}$ with the sheaf of sections $\mathscr{E}_\Sigma$.

## Empirical model

An empirical model is a *compatible family*

$$e \in \prod_{(U,\lambda)\in\mathcal{O}_\Sigma} \mathscr{D}_{\mathbb{R}^+}\mathscr{E}_\Sigma((U,\lambda))$$

the marginalisations of the distributions on different lowersets agree on intersections.

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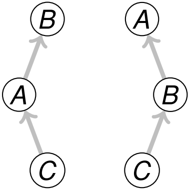
*The local states of empirical models are independent of the choice of foliation.*

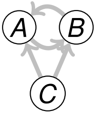
# From poset to preorders

By considering the entire lattice we expand our framework to include spaces that can accomodate empirical models compatible with certain classes of ‘indefinite causality’:



definite causality, posets

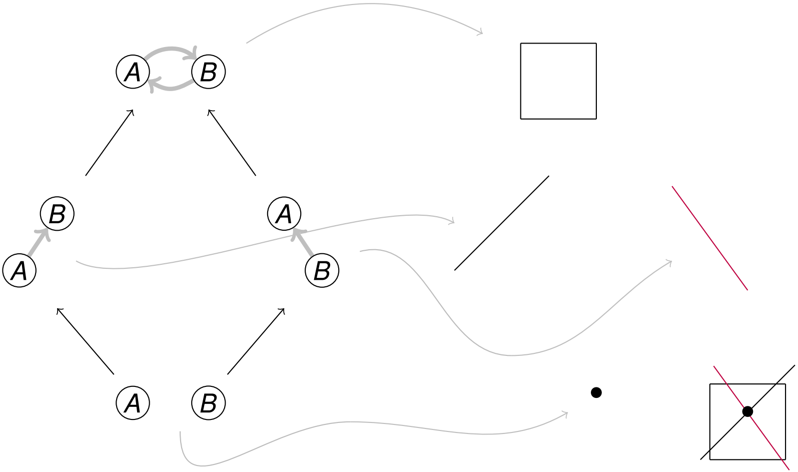
$\Longrightarrow$



indefinite causality, preorders

# The lattice of causal orders

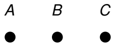
If we relax the notion of causal graphs to include also preorders, one can speak about the complete lattice of causal orders:



## Indefinite causal scenario

A *causal scenario* is the same as a definite causal scenario $\Sigma := (\Omega, \underline{I}, \underline{O}, \mathcal{I})$ except that $\Omega$ is now allowed to be a finite preorder (instead of a partial order). We say that a causal scenario is *definite* if $x \neq y$ implies $x \not\simeq y$ for all $x, y \in \Omega$ and we say that it is *indefinite* otherwise.

Suppose that we are considering three agents:



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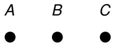

|  |  |  | $A_o$ | 0 | 0 | 1 | 1 |
|---|---|---|---|---|---|---|---|
|  |  |  | $B_o$ | 0 | 1 | 0 | 1 |
| $A_i$ | $B_i$ | $C_i$ | $C_o$ | * | * | * | * |
| 0 | 0 | 0 |  | 0.5 | 0.5 | 0.0 | 0.0 |
| 0 | 0 | 1 |  | 0.5 | 0.0 | 0.5 | 0.0 |
| 0 | 1 | 0 |  | 0.0 | 0.0 | 0.5 | 0.5 |
| 0 | 1 | 1 |  | 0.5 | 0.0 | 0.5 | 0.0 |
| 1 | 0 | 0 |  | 0.5 | 0.5 | 0.0 | 0.0 |
| 1 | 0 | 1 |  | 0.0 | 0.5 | 0.0 | 0.5 |
| 1 | 1 | 0 |  | 0.0 | 0.0 | 0.5 | 0.5 |
| 1 | 1 | 1 |  | 0.0 | 0.5 | 0.0 | 0.5 |

The table above represent a possible observed empirical scenario. What can be concluded abut the causal relations between the agents?

We can compute the causal decomposition of the scenario given in the previous slides:

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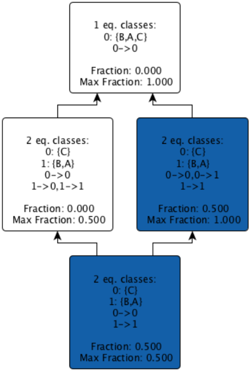

1 eq. classes:
0: {B,A,C}
0->0

Fraction: 0.000
Max Fraction: 1.000

2 eq. classes:
0: {C}
1: {B,A}
0->0
1->0,1->1

Fraction: 0.000
Max Fraction: 0.500

2 eq. classes:
0: {C}
1: {B,A}
0->0,0->1
1->1

Fraction: 0.500
Max Fraction: 1.000

2 eq. classes:
0: {C}
1: {B,A}
0->0
1->1

Fraction: 0.500
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The agents do not describe a definite causal scenario, the empirical model is not ‘supported’ by posets.

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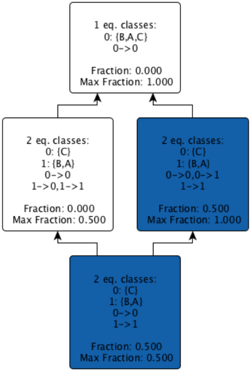

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1->0,1->1

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Max Fraction: 0.500

2 eq. classes:
0: {C}
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0->0,0->1
1->1

Fraction: 0.500
Max Fraction: 1.000

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1: {B,A}
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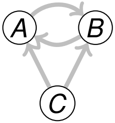
1->1

Fraction: 0.500
Max Fraction: 0.500



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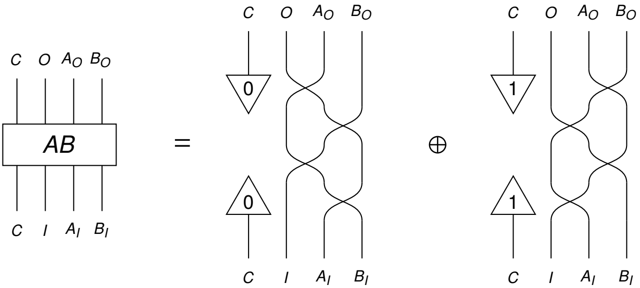
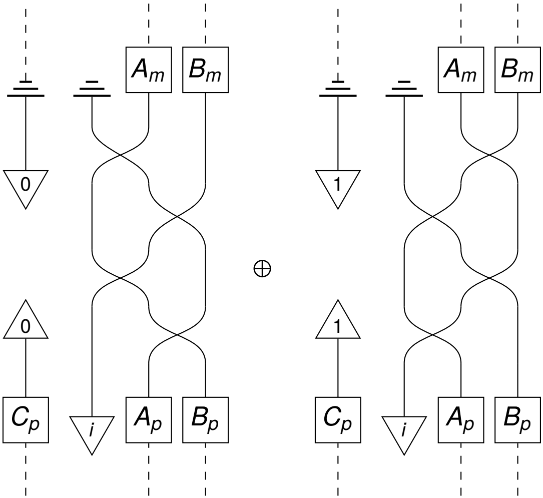
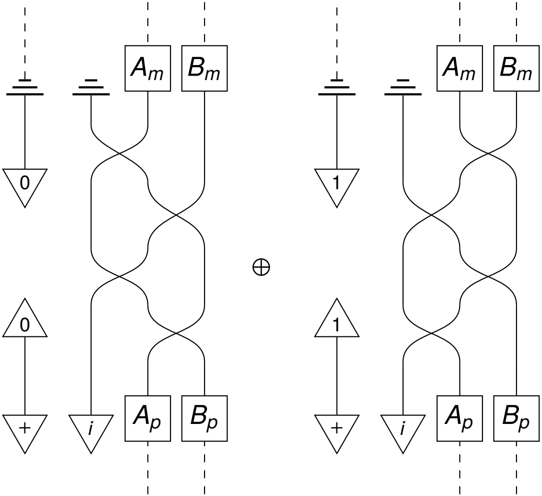
# Where is the empirical model coming from:



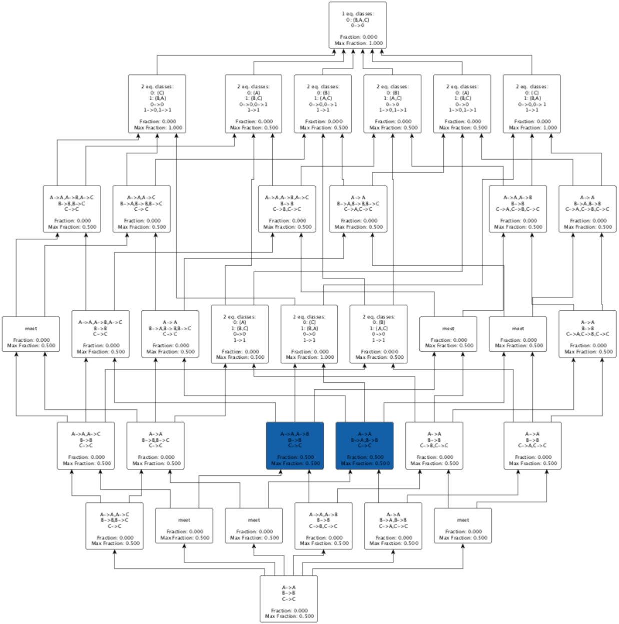

$A_m$
$B_m$
0
0
$C_p$
$i$
$A_p$
$B_p$
$\oplus$
$A_m$
$B_m$
1
1
$C_p$
$i$
$A_p$
$B_p$



A_m
B_m
0
0
+
i
A_p
B_p
⊕
A_m
B_m
1
1
+
i
A_p
B_p

|  |  |  | 0 | 0 | 1 | 1 |
|---|---|---|---|---|---|---|
|  |  |  | 0 | 1 | 0 | 1 |
|  |  |  | * | * | * | * |
| 0 | 0 | + | 0.5 | 0.25 | 0.25 | 0.0 |
| 0 | 1 | + | 0.25 | 0.0 | 0.5 | 0.25 |
| 1 | 0 | + | 0.25 | 0.5 | 0.0 | 0.25 |
| 1 | 1 | + | 0.0 | 0.25 | 0.25 | 0.5 |

# Probabilistic mixture of causal orders



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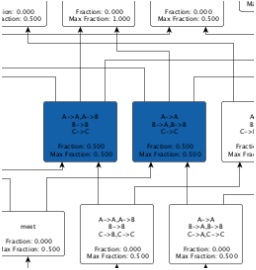
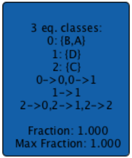
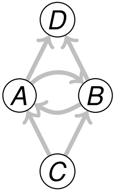


# The quantum switch



```
3 eq. classes:
0: {B,A}
1: {D}
2: {C}
0->0,0->1
1->1
2->0,2->1,2->2

Fraction: 1.000
Max Fraction: 1.000
```



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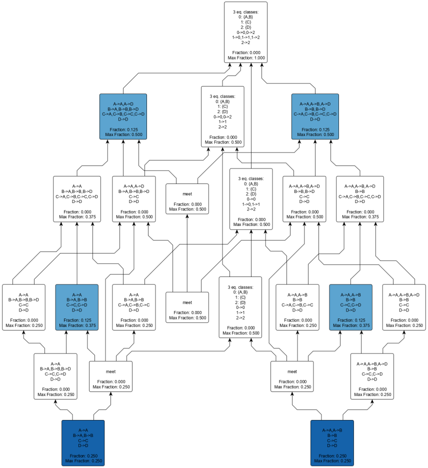

# A random empirical model